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Convergence of Operators and Korovkin's Theorem[1]

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I. INTRODUCTION AND NOTATION

Let L_n denote a sequence of operators defined on $C[0,1]$ ($C_{2\pi}$, respectively). The following, now a classical result, was proved by P. P. Korovkin [6].

Korovkin's Monotone Operator Theorem. If

(i) *for each n, L_n is a nonnegative operator,*

(ii) *$L_n p$ converges to p for the three functions $p = 1, x$ and x^2 ($1, \cos x, \sin x$, respectively), and*

(iii) $\lim_{n\to\infty} \|L_n\| = 1$,

then $L_n f$ converges to f for each f in $C[0,1]$ ($C_{2\pi}$, respectively).

Obviously (i) and (ii) imply (iii). Property (iii) has been included in the statement of the theorem only to facilitate a comparison to results proved below.

The purpose of this paper is to establish Korovkin-type theorems without assuming the existence of a lattice structure for the normed linear space which is the domain of the operators. Hence we must remove the monotonicity assumption on the operators. For example, the Korovkin Monotone Operator Theorem remains valid if we delete hypothesis (i) in the above statement. Perhaps the most interesting result proved here is the following

THEOREM. *Let L_n be a sequence of operators defined on $L^1[0,1]$. If*

(i) *$L_n 1$ converges to 1,*

(ii) *$L_n p$ converges weakly to p for the two functions $p = x$ and $p = x^2$, and*

(iii) $\lim_{n\to\infty} \|L_n\| = 1$,

then $L_n f$ converges to f for all f in $L^1[0,1]$.

In Section II we define a boundary for a subspace of a normed linear space, and prove a general convergence lemma. The classical Korovkin theorem is strengthened in Corollary 2. The complex analog of the Korovkin theorem

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is indicated in Corollary 4. Lemma 1 also generalizes a theorem on the convergence of resolvents which was proved by D. Ray (see [8], p. 44). Korovkin-type theorems for operators defined on $L^1[0,1]$ are proved in Section III. In Section IV we prove a result concerning the nonexistence of norm-one projections onto subspaces of finite codimension in $C(X)$. Several open questions are stated in the paper. For other generalizations of Korovkin's theorem (all requiring positive operators) see [3], [7], [11], [13] and the references cited there.

We use the remainder of this section to record the notation we shall use. If E is a normed linear space, E^* will denote the dual of E; and $S(E)$, the unit ball of E. If K is a convex set in a normed linear space, $\operatorname{ext} K$ is the set of extreme points of K. We identify the dual of $L^1[0,1]$ with $L^\infty[0,1]$. If E and F are two subspaces in duality, the $w(E,F)$-topology is the weak topology on E induced by F. For a compact Hausdorff space X, $C(X)$ is the Banach space of all real-valued, continuous functions on X, topologized by the supremum norm. $C_{2\pi}$ is the space of continuous, real-valued, 2π periodic functions on the real line. If f is a real function on a set X, let

$$(\operatorname{sgn} f)(x) = \begin{cases} 1, & \text{if } f(x) > 0, \\ 0, & \text{if } f(x) = 0, \\ -1, & \text{if } f(x) < 0. \end{cases}$$

II. A Fundamental Lemma. Operators on $C(X)$

Let P be a linear subspace of a normed linear space E. If L is an extreme point of $S(P^*)$, an elementary argument shows that there is at least one member of $\operatorname{ext} S(E^*)$ whose restriction to P is L (see, for example, [12], pp. 53–54). We call P *weakly separating* in E, if for each L in $\operatorname{ext} S(P^*)$, there is a unique member of $\operatorname{ext} S(E^*)$ which agrees with L on P. If P is weakly separating in E, we call the following subset of $\operatorname{ext} S(E^*)$ the *generalized Choquet boundary of* P (with respect to E):

$$cb(P) = \{F \text{ in } \operatorname{ext} S(E^*) \colon F \text{ restricted to } P \text{ is in } \operatorname{ext} S(P^*)\}.$$

If E is the space of real-valued continuous functions on a compact Hausdorff space X, and P is a linear subspace of E which contains the constants and separates the points of X, then $cb(P)$ consists of the evaluation functionals, and the negative of the evaluation functionals, of points in the classical Choquet boundary of P. For basic results related to classical Choquet boundaries and the generalized Choquet boundary, see, respectively, [8] and [14].

Lemma 1. *Let P be a weakly separating subspace of E. Let M be a linear subspace of E which contains P. Let L_n be a net of norm-one operators which carry M into E. Let f be in M.*

(i) *If $L_n(p)$ converges weakly to p for all p in P, then $k \circ L_n(f)$ converges to $k(f)$ for all k in $cb(P)$.*

(ii) *If $L_n(p)$ converges to p in norm, for all p in P, then $L_n(f)$ converges to f uniformly on $w(E^*, E)$-compact subsets of $cb(P)$.*

Proof. To prove (i), let k be in $cb(P)$. Let $L_i(f)$ be an arbitrary subnet of $L_n(f)$. It suffices to show that $L_i(f)$ contains a further subnet, say $L_j(f)$, such that $k \circ L_j(f)$ converges to $k(f)$. But, indeed, since $S(M^*)$ is $w(M^*, M)$-compact, L_i does admit a subnet L_j for which $k \circ L_j$ converges in the $w(M^*, M)$-topology. We denote the limit functional by H. From the hypotheses of the lemma, H is a norm-one functional on M which agrees with k on P. Since P is weakly separating in M, H and k agree on M. This proves the first part of the lemma.

Let K be a compact subset of $cb(P)$, and let f be in M. If $L_n(f)$ does not converge to f uniformly on K, there must exist an $r > 0$, a subnet L_j of L_n and points k_j in K, such that for all j

$$|k_j \circ L_j(f) - k_j(f)| \geqslant r.$$

Each functional $k_j \circ L_j$ has an extension to a functional in $S(E^*)$, say H_j. Using the compactness of K and $S(E^*)$ (in the $w(E^*, E)$-topology), we may choose a further subnet, say H_i, such that both k_i and H_i converge to, say, k and H, respectively. Now, for p in P,

$$|k(p) - H(p)| \leqslant |k(p) - k_i(p)| + |k_i(p) - H_i(p)| + |H_i(p) - H(p)|.$$

Since $L_i(p)$ converges to p in norm, $|k_i(p) - H_i(p)| = |k_i \circ (I - L_i)(p)|$ converges to zero. It follows that k and H agree on P. Since P is weakly separating in E, $k = H$. Hence, $H_i(h)$ converges to $k(h)$ for all h in E. Since

$$|k_i \circ L_i(f) - k_i(f)| \leqslant |k_i \circ L_i(f) - k(f)| + |k_i(f) - k(f)|,$$

and both terms on the right of the inequality approach zero, we have arrived at a contradiction. The lemma is proved.

We do not know appropriate conditions on the operators which would mply that $L_n f$ converges to f uniformly on $cb(P)$ for each f.

In the following corollary, let X be a compact Hausdorff space. Let P be a linear subspace of $C(X)$ which contains the constants and separates the points of X.

COROLLARY 2. *Let M be a linear subspace of $C(X)$ which contains P. Let L_n be a sequence of norm-one operators from M onto $C(X)$. Suppose the Choquet boundary (classical) of P is X. If $L_n(p)$ converges (converges weakly) to p for all p in P, then $L_n(f)$ converges (converges weakly) to f for all f in M.*

COROLLARY 3. *Let L_n be a sequence of norm-one operators defined on $C[0,1]$ ($C_{2\pi}$, respectively). Then $L_n(f)$ converges to f for all f in $C[0,1]$ ($C_{2\pi}$), if and only if $L_n(p)$ converges to p for the three functions 1, x and x^2 ($1, \cos x$ and $\sin x$, respectively).*

Proof. The necessity is obvious. The sufficiency follows from the last corollary and the fact that the Choquet boundary of the space spanned by 1, x and x^2 is all of $[0,1]$. Also $C_{2\pi}$ is isometrically isomorphic to the space of continuous functions defined on the circle in 2-space, and the Choquet boundary of the space spanned by 1, $\cos x$ and $\sin x$ is the entire circle.

Although throughout this paper we restrict our attention to real normed linear spaces, the method applies analogously to complex spaces. For example, making the obvious changes in the preceding argument, we easily establish the following

COROLLARY 4. *Let M be a linear subspace of continuous complex-valued functions defined on the unit circle (unit sphere respectively). Let M contain 1, z and $\bar{z}$ ($1, z, \bar{z}, z^2$ and $\bar{z}^2$ resp.). Let L_n be a sequence of norm operators which carry M into the space of continuous complex functions on the unit circle (sphere resp.). If $L_n(p)$ converges to p for $p = 1, z$ and $\bar{z}$ ($p = 1, z, \bar{z}, z^2$ and $\bar{z}^2$ resp.), then $L_n(f)$ converges to f for all f in M.*

We shall later need the following lemma which is proved in [*14*].

LEMMA 5. *Let P be a weakly separating subspace of a normed linear space E. The weak topology on $cb(P)$ induced by P is equivalent to the weak topology induced on $cb(P)$ by E.*

III. OPERATORS ON $L^1[0,1]$

We are particularly interested in proving Korovkin-type theorems for operators defined on $L^1[0,1]$. The desired theorem results easily for weakly separating subspaces, P, of $L^1[0,1]$ such that $cb(P) = \operatorname{ext} S(L^1[0,1]^*)$. However, it is known [*9*] that if P is finite dimensional, there is a k in $\operatorname{ext} S(L^1[0,1]^*)$ such that $k(p) = 0$ for all p in P. Hence, this approach cannot work.

Let P be a reflexive subspace of $L^1[0,1]$ such that 0 is the only member of P that vanishes on a set of positive measure.

Let

$$K = \{\operatorname{sgn} f : f \text{ in } P\}.$$

LEMMA 6. *Every functional in P^* has a unique norm-preserving extension to a member of $L^1[0,1]^*$ (hence P is weakly separating), and $cb(P) = K$.*

Proof. Let k be a norm-one functional in P^*. Now, k has an extension to a member of $S(L^1[0,1]^*)$, say, k' (we identify $L^1[0,1]^*$ with $L^\infty[0,1]$). Since k' has norm-one over P, and P is reflexive, there is a $p \neq 0$ in P such that

$$\int_0^1 pk'\,dx = \|p\| = \int_0^1 p \operatorname{sgn} p\,dx.$$

Since k' has norm-one, $(\operatorname{sgn} p) - k'$ is nonnegative on $\{x: p(x) > 0\}$ and nonpositive on $\{x: p(x) < 0\}$. Hence if $\operatorname{sgn} p \neq k'$

$$\int_0^1 p(\operatorname{sgn} p - k')\,dx > 0.$$

Therefore, $\operatorname{sgn} p = k'$.

COROLLARY 7. *If P is a finite dimensional subspace of $L^1[0,1]$ spanned by polynomials or by trigonometric polynomials, then P is weakly separating in $L^1[0,1]$, and*

$$cb(P) = \{\operatorname{sgn} f : f \text{ in } P\}.$$

COROLLARY 8. *Let P be a reflexive subspace of $L^1[0,1]$ for which 0 is the only member of P which vanishes on a set of positive measure. Then*

$$P^\perp = \left\{f \text{ in } L^\infty[0,1]: \int_0^1 p(x)\, f(x)\,dx = 0\right\},$$

for all p in P, is a Chebyshev subspace of $L^\infty[0,1]$.

The last corollary is an immediate consequence of Lemma 6 and of a theorem of R. R. Phelps [*9*]. In particular, this corollary provides a method for generating Chebyshev subspaces of finite codimension in L^∞. The only other known Chebyshev subspaces of finite codimension in spaces of the type $C(X)$ seem to be those constructed by Garkavi [*4*] and Phelps [*10*]. It is unknown if there are any Chebyshev subspaces of infinite dimension and infinite codimension in any $C(X)$. Clearly, the existence of an infinite dimensional subspaces of $L^1[0,1]$ satisfying the conditions of the corollary would settle the problem in the form stated here.

Note added in proof. Professor J. Lindenstrauss has brought an example to my attention which solves this problem. For every $1 < p < 2$ there is a subspace E of $L^1[0,1]$ which is isometric to $L^p[0,1]$ (see e.g. Lindenstrauss and Pelczynski, *Studia Math.* XXIX (1968), page 311, Corollary 1). Since E is a smooth Banach space, the nonzero functions in E have precisely the same support. For suppose f and g are in E then define $p = \operatorname{sgn} f$, and

$$q(x) = \begin{cases} \operatorname{sgn} f(x): x \text{ in } (\text{support} f) \\ \operatorname{sgn} g(x): \text{otherwise}, \end{cases}$$

If (support g) – (support f) has positive measure, p and q would generate distinct hyperplanes which support $S(E)$ at f. Since E is smooth this is not possible. Let F denote the common support of the members of E. Then, in the obvious way, E is a subspace of $L^1(F)$ with the desired properties. Now $L^1(F)$ is isometric to $L^1[0,1]$ (Halmos, R. R., "Measure Theory," Van Nostrand, Princeton (1950), page 173). Let E' denote the image in $L^1[0,2]$ of E under this isometry. The fact that the zero function is the only member of E' vanishing on a set of positive measure is immediate from the corresponding property in $E \subseteq L^1(F)$, and the observation that two summable functions f and g have disjoint supports if and only if $\|f+g\| = \|f\| + \|g\| = \|f-g\|$.

In the following, let P denote the subspace of $L^1[0,1]$ spanned by 1, x and x^2 or by 1, $\cos x$ and $\sin x$. Let L_n be a sequence of norm-one operators which carry $L^1[0,1]$ into itself.

THEOREM 9. *$L_n f$ converges to f, for each f in $L^1[0,1]$, if (and obviously only if) the following conditions are satisfied:*

(i) *$L_n 1$ converges to 1, and*
(ii) *$L_n p$ converges weakly to p for each p in P.*

Proof. Let K denote the subset of $L^1[0,1]^*$

$\{g$ in $L^\infty[0,1]$: g is the characteristic function of a subinterval of $[0,1]\}$.

Since every member of K can be written as the average of two members of $cb(P)$ (Corollary 7), we know that $k \circ L_n(f)$ converges to $k(f)$ for each k in K and each f in $L^1[0,1]$ (Lemma 1).

Let

$G = \{g$ in $L^1[0,1]$: g is the characteristic function of a subinterval of $[0,1]$, or is the characteristic function of the complement in $[0,1]$ of such a subinterval$\}$.

Since the operators L_n are bounded, and G is fundamental in $L^1[0,1]$, it suffices to show that $L_n g$ converges to g for each g in G.

Let g be in G. We know that $k \circ L_n(g)$ converges to $k(g)$ for each k in K. Let

$$L_n'(g)(x) = \begin{cases} L_n(g)(x) & \text{if } g(x) \neq 0, \\ 0 & \text{if } g(x) = 0. \end{cases}$$

Now $k(L_n'(g))$ converges to $k(g)$, for each k in K. Hence,

$$\lim_{n\to\infty} \|L_n'(g)\| \geqslant \|g\|.$$

For, certainly if this were not true,

$$\lim_{n\to\infty} \int_0^1 L_n'(g)(x)\cdot 1\, dx \leqslant \lim_{n\to\infty} \int_0^1 |L_n'(g)(x)|\, dx = \lim_{n\to\infty} \|L_n'(g)\| < \|g\| = \int_0^1 g(x)\cdot 1\, dx.$$

Since 1 is in K, the above inequality cannot arise.

Let

$$Z = \{x \text{ in } [0,1] \colon g(x) = 0\}.$$

From the above we have that if

$$\lim_{n\to\infty} \int_Z |L_n(g)(x)|\, dx > 0,$$

then

$$\lim_{n\to\infty} \|L_n(g)\| > \|g\|.$$

This, in turn, implies that $\|L_n\| > 1$ for sufficiently large n. Since this is not possible,

$$\lim_{n\to\infty} \int_Z |L_n(g)(x)|\, dx = 0.$$

Now let $f = 1 - g$. Since f is in G, we know that

$$\lim_{n\to\infty} \int_{[0,1]-Z} |L_n(f)(x)|\, dx = 0.$$

Since $L_n(f) = L_n(1) - L_n(g)$, and $L_n 1$ converges to 1, we conclude that

$$\lim_{n\to\infty} \int_{[0,1]-Z} |1 - L_n(g)x|\, dx = 0.$$

Hence, $L_n(g)$ converges to g, as we wished to show. This completes the proof.

Remark. The crucial properties of the subspace P in the theorem are that P is a 3-dimensional Haar subspace (see [*1*] for definition and basic properties) and that the linear span of $cb(P)$ contains the characteristic function of every closed interval. Without altering the proof, the theorem applies to any subspace P with the latter properties, and hence to any 3-dimensional Haar subspace which contains a strictly positive function.

In the preceding theorem we assumed that the domain of the operators was all of $L^1[0,1]$. We do not know if the theorem is true when the domain is an arbitrary linear subspace of $L^1[0,1]$ which contains P. For example, suppose f is not in P. Let M denote the space spanned by $\{f\} \cup P$, and assume L_n is a sequence of norm-one operators which carry M into $L^1[0,1]$. Is it true that if $L_n(p)$ converges to p for all p in P, then $L_n f$ converges to f?

We can establish the following

PROPOSITION 10. *If M and L_n are as described above, then $L_n f$ converges to f uniformly on*

$$\{g \text{ in } L^1[0,1]^* : g \text{ is the characteristic function of a subinterval of } [0,1]\}.$$

Proof. A proof is easily constructed from Lemma 1, Corollary 7, and Lemma 12 below.

LEMMA 11. *Let P be a linear subspace of $L^1[0,1]$ for which 0 is the only member of P vanishing on a set of positive measure. Then P has a smooth norm.*

Proof. This lemma is a consequence of the proof of Lemma 6. We also note that Lemma 11 generalizes to any measure space.

LEMMA 12. *Let P be a finite dimensional subspace of $L^1[0,1]$ such that 0 is the only member of P vanishing on a set of positive measure. Then $cb(P)$ is $w(L^1[0,1]^*, L^1[0,1])$-compact.*

Proof. Let k_i be a net of functionals in $cb(P)$. Let k_i' denote the restriction of k_i to P. Since P is finite dimensional, we may assume that k_i' converges to some functional, say k', in norm. Since P^* is strictly convex by Lemma 11, and k' has norm one, k' is an extreme point of $S(P^*)$. By Lemma 6, P is weakly separating. Hence, there is a unique extreme point k in $S(L^1[0,1]^*)$ which agrees with k' on P. We know that k_i converges to k in the $w(L^1[0,1]^*, P)$-topology. But by Lemma 5, this implies that k_i converges to k in the $w(L^1[0,1]^*, L^1[0,1])$-topology. This completes the proof.

IV. NONEXISTENCE OF NORM-ONE PROJECTIONS

Let P denote the linear space spanned by 1, x, and x^2. From the classical theorems concerning projections in Hilbert space we know that there is a projection of norm one of $L^2[0,1]$ onto P. However, as a consequence of Proposition 10 and Lemma 1, we essentially have the opposite situation in $L^1[0,1]$ and $C[0,1]$. We state this for $L^1[0,1]$.

COROLLARY 13. *Let f belong to $L^1[0,1]$. The identity operator is the only norm-one operator which carries $\{f\} \times P$ into $L^1[0,1]$ and acts as the identity on P (here $\{f\} \times P$ denotes the subspace of $L^1[0,1]$ spanned by $\{f\} \cup P$).*

We conclude with a proposition related to the last corollary.

PROPOSITION 14. *Let X be a compact Hausdorff space which contains at most a finite number of isolated points: $x_1, x_2, \ldots, x_n$. Let P be a subspace of finite codimension in $C(X)$, and let L be a norm-one linear operator defined on $C(X)$ which acts as the identity on P. Then for each f in $C(X)$ and each nonisolated y in X, $(Lf)(y) = f(y)$.*

Proof. Let $Y = X - \{x_1, x_2, \ldots, x_n\}$, and let m be the codimension of P. For a point x in X, let $e(x)$ denote the point evaluation functional of x. Let

$$E = \{x \text{ in } X\text{: the restriction of } e(x) \text{ to } P \text{ is in } \operatorname{ext} S(P^*)\}.$$

We first show that the closure of E contains Y. If this were not true, there would exist a nonempty open set U in Y which does not intersect E. We can construct $m+1$ continuous functions, f_i, all of norm one, which have disjoint supports and such that the support of each is contained in U. Since P has codimension m, there must exist a nontrivial linear combination of the functions $f_1, f_2, \ldots, f_{m+1}$ which is in P. By our construction of the functions f_i, this linear combination f is a nontrivial function which vanishes off U.

Since every extreme point of $S(P^*)$ agrees with some point evaluation functional on P, we have that $k(f) = 0$ for each k in $\operatorname{ext} S(P^*)$. But

$$\|f\| = \max\{|k(f)| : k \text{ is in } \operatorname{ext} S(P^*)\},$$

thus contradicting the fact that f is not identically zero.

Now let

$$E' = \{x \text{ in } E\text{: if } y \text{ is in } X - x, \text{ there is a } p \text{ and a } q \text{ in } P, \text{ for which } p(x) \neq p(y) \text{ and } q(x) \neq -q(y)\}.$$

Since P has finite codimension, all but a finite number of points in E are in E' (otherwise we could construct $m+1$ linearly independent functionals in the annihilator of P). It follows that the closure of E' contains Y. We observe that the functionals in $S(C(X)^*)$ whose restrictions to P agree with some specified extreme point in $S(P^*)$, form an extremal subset of $S(C(x)^*)$. Hence, if x is in E', h is in $S(C(X)^*)$, and $p(x) - h(p) = 0$ for all p in P, then $e(x) = h$.

Consider now the norm-one operator L in the statement of the proposition. We see that $e(x) \circ L$ is in $S(C(X)^*)$, and $e(x) \circ L(p) - p(x) = 0$ for all p in P. Thus, $(Lf)(x) = f(x)$ for all x in E'. Since Lf is a continuous function, and the closure of E' contains Y, we conclude that $(Lf)(y) = f(y)$ for all y in Y.

COROLLARY 14. *Let X be a compact Hausdorff space. If there exists a norm-one projection of $C(X)$ onto a subspace of finite codimension n, then X contains n isolated points.*

P. D. Morris has communicated an independent proof of the last corollary. His proof uses an interesting technique of solving the dual problem concerning the existence of continuous linear metric selections associated with finite dimensional subspaces of $C(X)$.

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